M3 May 13 2015

1. A particle P of mass 0.5 kg is attached to one end of a light elastic spring, of natural length 1.2 m and modulus of elasticity λ newtons. The other end of the spring is attached to a fixed point A on a ceiling. The particle is hanging freely in equilibrium at a distance 1.5 m vertically below A.

(a) Find the value of λ. **(3)**

The particle is now raised to the point B, where B is vertically below A and $AB = 0.8$ m. The spring remains straight. The particle is released from rest and first comes to instantaneous rest at the point C.

(b) Find the distance AC. **(4)**

1a) $R(\uparrow)$: $T - 0.5g = 0$

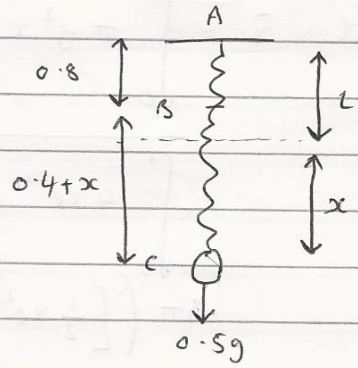
$T = 0.5g$

using Hookes law:

$T = \frac{\lambda x}{l}$

$0.5g = \frac{\lambda (0.3)}{1.2}$

$\lambda = 19.6$

1b 

At B

$E.E = \frac{\lambda x^2}{2l}$

$= \frac{19.6(0.4)^2}{2(1.2)}$

$= \frac{98}{75}$

$P.E = mgh$

$= 0.5g(0.4+x)$

$= 0.2g + 0.5gx$

$KE = 0$

At C

$E.E = \frac{\lambda x^2}{2l}$

$= \frac{19.6x^2}{2(1.2)}$

$= \frac{49}{6}x^2$

$KE = 0.$

$PE = 0$

using conservation of energy:

$\frac{98}{75} + 0.2g + 0.5gx = \frac{49}{6}x^2$

$0 = \frac{49}{6}x^2 - 4.9x - \frac{49}{15}$

$x = 1 \quad x = -\frac{2}{5}$ (reject)

$\therefore AC = 1.2 + 1$

$= 2.2$ m

2. The finite region bounded by the x-axis, the curve with equation $y = 2e^x$, the y-axis and the line $x = 1$ is rotated through one complete revolution about the x-axis to form a uniform solid.

Use algebraic integration to

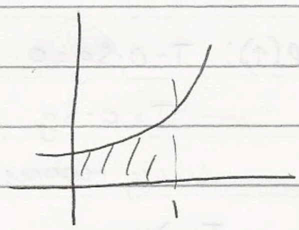
(a) show that the volume of the solid is $2\pi(e^2 - 1)$,

(4)

(b) find, in terms of e, the x coordinate of the centre of mass of the solid.

(6)

2a. $\text{Volume} = \int_0^1 4e^{2x} \pi \, dx$



$= \pi \left[2e^{2x}\right]_0^1$

$= \pi \left[2e^2 - 2e^0\right]$

$= 2\pi \left[e^2 - 1\right]$

2b. $\text{vol} \times \bar{x} = \int_0^1 \pi y^2 x \, dx$

Let $u = x$, $\frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{2x}$, $v = \frac{1}{2}e^{2x}$

$= \pi \int_0^1 4xe^{2x} \, dx$

$= 4\pi \left(\left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x} \, dx \right)$

$= 4\pi \left(\frac{1}{2}e^2 - \frac{1}{4}\left[e^{2x}\right]_0^1 \, dx \right)$

$= 4\pi \left(\frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1) \right)$

$= 4\pi \left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} \right)$

$= \pi (e^2 + 1)$

$\bar{x} = \dfrac{\pi (e^2 + 1)}{2\pi (e^2 - 1)}$

$= \dfrac{e^2 + 1}{2(e^2 - 1)}$

3.

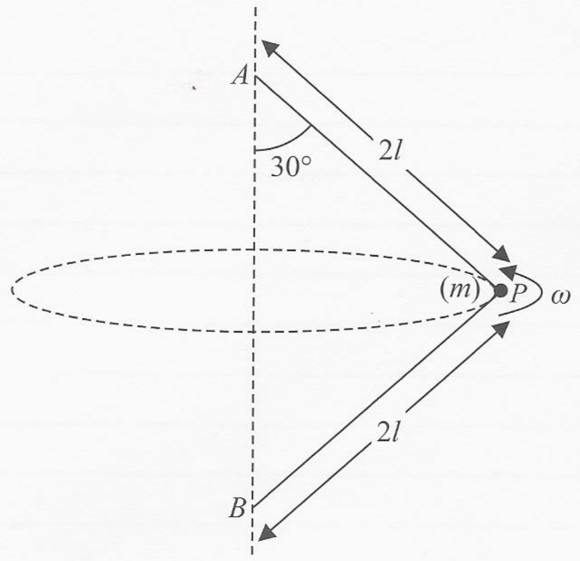


Figure 1

A small ball P of mass m is attached to the midpoint of a light inextensible string of length $4l$. The ends of the string are attached to fixed points A and B, where A is vertically above B. Both strings are taut and AP makes an angle of 30° with AB, as shown in Figure 1. The ball is moving in a horizontal circle with constant angular speed ω.

(a) Find, in terms of m, g, l and ω,

(i) the tension in AP,

(ii) the tension in BP.

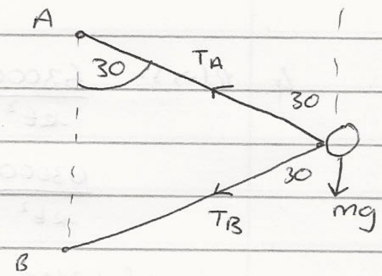
(8)

(b) Show that $\omega^2 \geqslant \dfrac{g\sqrt{3}}{3l}$.

(2)

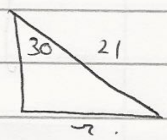
3:- Let Tension in AP = T_A

Tension in BP = T_B.



R(↑): $T_A \cos 30 = mg + T_B \cos 30$

$T_A\left(\frac{\sqrt{3}}{2}\right) = mg + T_B\left(\frac{\sqrt{3}}{2}\right)$ (2)



R(←): $T_A \sin 30 + T_B \sin 30 = m(r\omega^2)$

$\frac{1}{2} T_A + \frac{1}{2} T_B = ml\omega^2$

$T_A = 2ml\omega^2 - T_B$ (1)

$\therefore \sin 30 = \frac{r}{2l}$.

$r = l$.

Substitute (1) into (2)

$(2ml\omega^2 - T_B)\left(\frac{\sqrt{3}}{2}\right) = mg + T_B\frac{\sqrt{3}}{2}$.

$ml\omega^2\sqrt{3} = mg + T_B\sqrt{3}$.

$T_B = ml\omega^2 - \frac{mg}{\sqrt{3}}$

$$\therefore T_A = 2ml\omega^2 - \left(ml\omega^2 - \frac{mg}{\sqrt{3}}\right)$$
$$= ml\omega^2 + \frac{mg}{\sqrt{3}}$$

6. $T_B \geqslant 0$

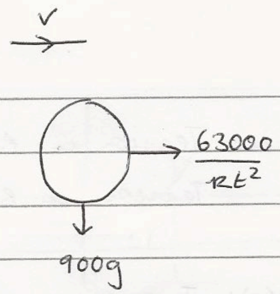
$$ml\omega^2 - \frac{mg}{\sqrt{3}} \geqslant 0$$
$$l\omega^2 \geqslant \frac{g}{\sqrt{3}}.$$
$$\omega^2 \geqslant \frac{g}{l\sqrt{3}}$$
$$\omega^2 \geqslant \frac{\sqrt{3}\,g}{3l}.$$

4. A vehicle of mass 900 kg moves along a straight horizontal road. At time t seconds the resultant force acting on the vehicle has magnitude $\frac{63000}{kt^2}$ N, where k is a positive constant. The force acts in the direction of motion of the vehicle. At time t seconds, $t \geqslant 1$, the speed of the vehicle is v m s^{-1} and the vehicle is a distance x metres from a fixed point O on the road. When $t = 1$ the vehicle is at rest at O and when $t = 4$ the speed of the vehicle is 10.5 m s^{-1}.

(a) Show that $v = 14 - \frac{14}{t}$
(7)

(b) Hence deduce that the speed of the vehicle never reaches 14 m s^{-1}.
(1)

(c) Use the trapezium rule, with 4 equal intervals, to estimate the value of x when $v = 7$
(4)



4. $R(\rightarrow)$: $\frac{63000}{kt^2} = 900a.$

$$\frac{63000}{kt^2} = 900\frac{dv}{dt}$$

$$\int \frac{63000}{kt^2}\,dt = \int 900\,dv.$$

$$\frac{-63000}{kt} = 900v + c$$

when $t=1$, $v=0$.

$$\frac{-63000}{k} = c.$$

$$\frac{-63000}{kt} = 900v - \frac{63000}{k}.$$

when $t = 4$, $v = 10.5$.

$$\frac{-15750}{R} = 900(10.5) - \frac{63000}{R}$$

$$\frac{47250}{R} = 9450$$

$$R = 5.$$

$$\therefore \frac{-63000}{5t} = 900v - \frac{63000}{5}$$

$$-\frac{12600}{t} = 900v - 12600$$

$$14 - \frac{14}{t} = v.$$

4b. As t is positive, for large values of T, the velocity is just below $14\ ms^{-1}$. It is an assymptote.

4c. when $v = 7$

$$14 - \frac{14}{t} = 7.$$

$$\therefore 7 = \frac{14}{t}.$$

$$t = 2.$$

$$\therefore \frac{2-1}{4} = 0.25.$$

t	1	1.25	1.50	1.75	2.00
v	0	2.8	$\frac{14}{3}$	6	7

$$\therefore x \approx \frac{1}{2}(0.25)\left(0 + 7 + 2\left(2.8 + \frac{14}{3} + 6\right)\right) \approx 4.24\ m\ (3sf)$$

5.

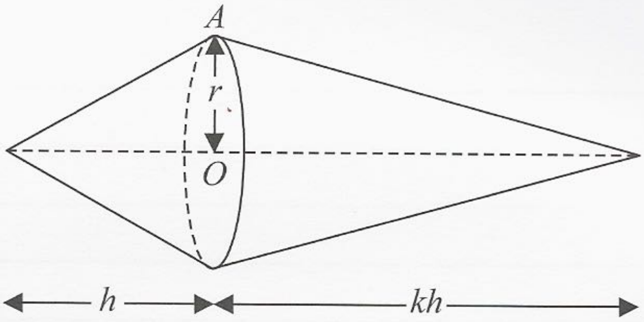


Figure 2

Figure 2 shows a uniform solid spindle which is made by joining together the circular faces of two right circular cones. The common circular face has radius r and centre O. The smaller cone has height h and the larger cone has height kh. The point A lies on the circumference of the common circular face. The spindle is suspended from A and hangs freely in equilibrium with AO at an angle of 30° to the vertical.

Show that $k = \dfrac{4r}{h\sqrt{3}} + 1$

(6)

5. Cone 1

mass: $\frac{1}{3}\pi r^2 h\rho$

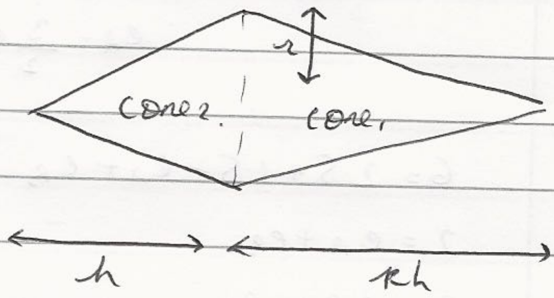
$= \frac{1}{3}\pi r^2 kh\rho$

Centre $= \frac{1}{4}h$

$= \frac{kh}{4}$

$= \left(\frac{kh}{4}, 0\right)$

Mass ratio $= k$



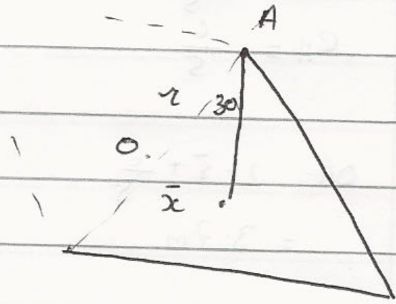
Cone 2

mass $= \frac{1}{3}\pi r^2 h\rho$

Centre $= \frac{1}{4}h$

$= \left(-\frac{1}{4}h, 0\right)$

mass ratio: 1.



$\therefore \tan 30 = \dfrac{h(k^2-1)}{4r(k+1)}$

$\dfrac{1}{\sqrt{3}} = \dfrac{h(k-1)}{4r}$

$\dfrac{4r}{h\sqrt{3}} + 1 = k.$

$\therefore k\left(\frac{kh}{4}\right) + \left(-\frac{1}{4}h\right)(1) = (k+1)(\bar{x})$

$\frac{k^2h}{4} - \frac{h}{4} = (k+1)\bar{x}$

$\bar{x} = \dfrac{h(k^2-1)}{4(k+1)}$

6.

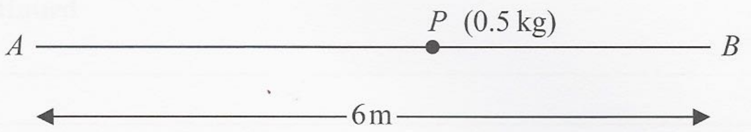


Figure 3

Two points A and B are 6 m apart on a smooth horizontal floor. A particle P of mass 0.5 kg is attached to one end of a light elastic spring, of natural length 2.5 m and modulus of elasticity 20 N. The other end of the spring is attached to A. A second light elastic spring, of natural length 1.5 m and modulus of elasticity 18 N, has one end attached to P and the other end attached to B, as shown in Figure 3. Initially P rests in equilibrium at the point O, where AOB is a straight line.

(a) Find the length of AO.

(4)

The particle P now receives an impulse of magnitude 6 N s acting in the direction OB and P starts to move towards B.

(b) Show that P moves with simple harmonic motion about O.

(4)

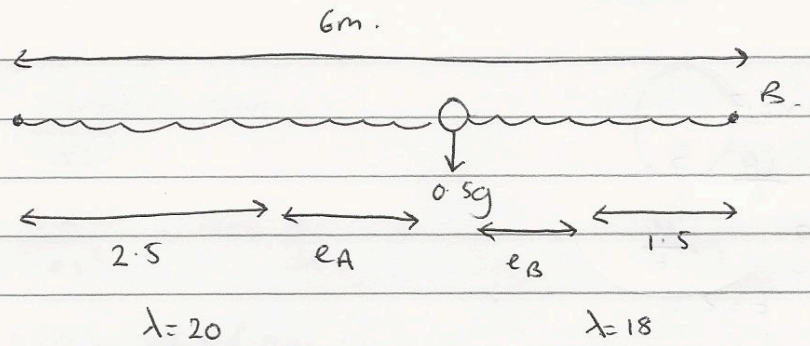
(c) Find the amplitude of the motion.

(4)

(d) Find the time taken by P to travel 1.2 m from O.

(3)

6.



Let extension of $AO = e_A$, extension of $OB = e_B$.

<u>For OA</u>

$T = \frac{\lambda x}{l}$

$= \frac{20}{2.5} e_A$

$= 8e_A$

<u>For OB</u>

$T = \frac{\lambda x}{l}$

$= \frac{18}{1.5} e_B$

$= 12e_B$

$R(\leftarrow)$: $8e_A = 12e_B$.

$\therefore e_A = \frac{3}{2} e_B$.

$6 = 2.5 + 1.5 + e_A + e_B$

$2 = e_A + e_B$

$2 = \frac{3}{2}e_B + e_B$

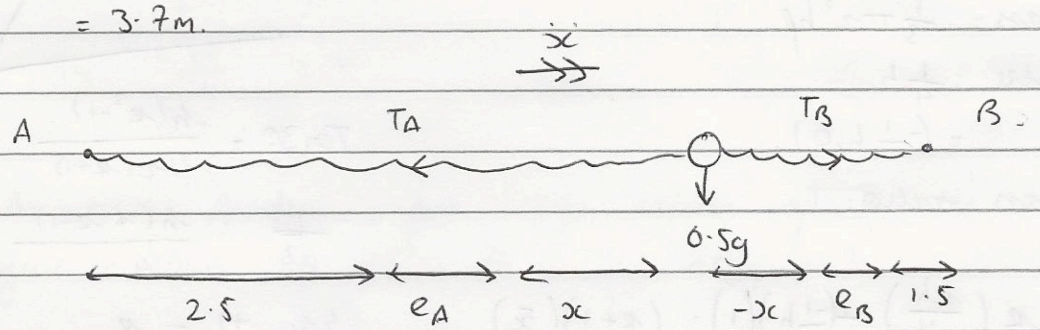
$e_B = \frac{4}{5}$

$e_A = \frac{6}{5}$

$OA = 2.5 + \frac{6}{5}$

$= 3.7m.$

66.



~~(a)~~ <u>For OA</u>

Using Hookes law: $T_A = \frac{\lambda x}{l}$

$= \frac{20(e_A + x)}{2.5}$

$= 8(e_A + x)$

<u>For OB</u>

Using Hookes law: $T_B = \frac{\lambda x}{l}$

$= \frac{18}{1.5}(e_B - x)$

$= 12(e_B - x)$

$R(\rightarrow)$: $12(e_B - x) - 8(e_A + x) = 0.5\ddot{x}$

$\frac{48}{5} - 12x - \frac{48}{5} - 8x = 0.5\ddot{x}$

$-40x = \ddot{x}$

$\therefore$ <u>SHM.</u>

c. $I = m(v-u)$.

$6 = 0.5(v-0)$

$v = 12\,ms^{-1}$. $\rightarrow$ max speed.

$\omega^2 = 40$

$\omega = 2\sqrt{10}$.

$v = a\omega$

$\frac{12}{2\sqrt{10}} = a$

$a = 1.897366$

$= 1.90$ (3 sf).

d. $x = a \sin \omega t$.

$1.2 = 1.90 \sin(\sqrt{40}\, t)$

$\sqrt{40}t = 0.68$

$t = 0.108$ seconds (3 sf).

7. A solid smooth sphere, with centre O and radius r, is fixed to a point A on a horizontal floor. A particle P is placed on the surface of the sphere at the point B, where B is vertically above A. The particle is projected horizontally from B with speed $\dfrac{\sqrt{gr}}{2}$ and starts to move on the surface of the sphere. When OP makes an angle θ with the upward vertical and P remains in contact with the sphere, the speed of P is v.

(a) Show that $v^2 = \dfrac{gr}{4}(9 - 8\cos\theta)$.

(4)

The particle leaves the surface of the sphere when $\theta = \alpha$.

(b) Find the value of $\cos\alpha$.

(4)

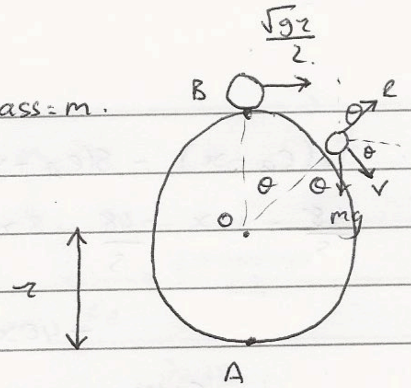
After leaving the surface of the sphere, P moves freely under gravity and hits the floor at the point C.

Given that $r = 0.5\,\text{m}$,

(c) find, to 2 significant figures, the distance AC.

(7)

Take A to have 0 PE. Let mass = m.



7. At B

$PE = mgh$

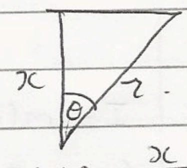
$= mg(2r)$

$= 2mgr$.

$KE = \frac{1}{2}mv^2$

$= \frac{1}{2}m\left(\frac{1}{4}gr\right)$

$= \frac{1}{8}mgr$.



$\therefore \cos\theta = \frac{x}{r}$

$x = r\cos\theta$.

At angle θ

$PE = mgh$.

$= mgr(1 + \cos\theta)$

$KE = \frac{1}{2}mv^2$

conservation of energy: $2mgr + \frac{1}{8}mgr = mgr + mgr\cos\theta + \frac{1}{2}mv^2$

$\frac{9}{8}gr - gr\cos\theta = \frac{1}{2}v^2$

$\frac{gr}{4}(9 - 8\cos\theta) = v^2$.

7b. R(↖): towards centre of sphere

$mg\cos\theta - R = m\left(\frac{v^2}{r}\right)$

$R = 0$.

$mg\cos\theta = m\left(\frac{gr}{4r}(9 - 8\cos\theta)\right)$

$mg\cos\theta = \frac{mg}{4}(9 - 8\cos\theta)$

$4\cos\theta = 9 - 8\cos\theta$.

$12\cos\theta = 9$

$\cos\theta = \frac{3}{4}$.

$\cos\alpha = \frac{3}{4}$.

7c. At angle α $v^2 = \frac{gr}{4}\left(9 - 8\left(\frac{3}{4}\right)\right)$

$v = \sqrt{\frac{3gr}{4}}$

vertical component : $v\sin\alpha = \sin\alpha\sqrt{\frac{3g}{8}}$

Height above ground $= r + r\left(\frac{3}{4}\right)$

$= \frac{7}{8}$ metres.

$s = ut + \frac{1}{2}at^2$.

$\frac{7}{8} = \left(\sin\alpha\sqrt{\frac{3g}{8}}\right)t + \frac{1}{2}g(t^2)$

$4.9t^2 + \left(\sin\alpha\sqrt{\frac{3g}{8}}\right)t - \frac{7}{8} = 0$

$t = \cancel{0.3141710}$ 0.3125544...

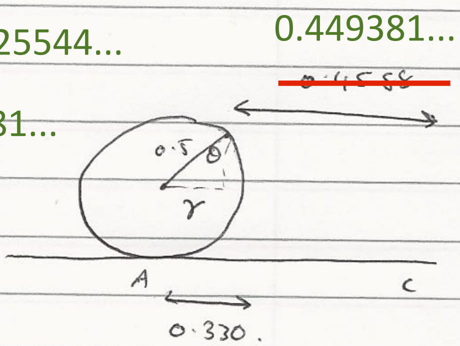
Horizontal component : $v\cos\alpha = 5\cos\alpha\sqrt{\frac{3g}{8}}$

distance travelled in time $t = \cancel{0.314...}$ 0.3125544...

$v = \frac{s}{t}$.

$\therefore s = \cos\alpha\sqrt{\frac{3g}{8}} \times \cancel{0.314}$ 0.3125544...

$= \cancel{0.4588...}$ 0.449381...



$\therefore \sin\theta = \frac{r}{0.5}$

$\therefore r = 0.330$

$\therefore$ From A to C $= \cancel{0.4588} + 0.330$.. 0.449381 + 0.330

$= \cancel{0.78961}$

$= \cancel{0.79\ m\ (2\ sf)}$. 0.779 (3 sf)